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On the Valuation and Analysis of Risky Debt: A Theoretical Approach Using a Multivariate Extension of the Merton Model

Edwin O. Fischer
Lisa-Maria Kampl
Ines Wöckl

University of Graz
Institute of Finance

34th Workshop of the AWG on Banking and Finance
Vaduz, 22.11.2019

Available at: http://dx.doi.org/10.2139/ssrn.3450524

Motivation

- The valuation and analysis of risky debt has been shaped by the **Merton Model (1974).**
- In the model the firm's equity at maturity can be interpreted as a European call option
  - written on the underlying asset V_T (value of the firm) that is distributed logarithmically normally
  - with the nominal value of the debt Nom as the exercise price and
  - maturity T.

$$E_T = \begin{Bmatrix} 0 & if\ V_T < Nom \\ V_T - Nom & if\ V_T \geq Nom \end{Bmatrix}$$

- The value of debt can be written as

$$D_T = \begin{Bmatrix} V_T & if\ V_T < Nom \\ Nom & if\ V_T \geq Nom \end{Bmatrix}$$

Motivation

Applying the **Black & Scholes** formula for European call options the firm's debt at time $t = 0$ can be expressed as

$$D_0 = V_0 \cdot [1 - N(d_1)] + Nom \cdot e^{-r \cdot T} \cdot N(d_2),$$

where

$$d_1 = \frac{ln\frac{V_0}{Nom} + \left(r + \frac{\sigma_V^2}{2}\right)T}{\sigma_V \cdot \sqrt{T}} \text{ and } d_2 = \frac{ln\frac{V_0}{Nom} + \left(r - \frac{\sigma_V^2}{2}\right)T}{\sigma_V \cdot \sqrt{T}} = d_1 - \sigma_V \cdot \sqrt{T},$$

and $N(.)$ is the cumulative standard normal distribution

Interpretation:

$$D_0 = Nom \cdot e^{-r \cdot T} - \left[Nom \cdot e^{-r \cdot T} - V_0 \cdot \frac{N(-d_1)}{N(-d_2)}\right] \cdot N(-d_2)$$

Motivation

Limitations of the Standard Merton Model

Constant interest rates	Firm can only default at its debt maturity date T	Only applicable for valuing zero-coupon debt instruments	Only applicable if firm's debt consists of only one single debt instrument

- We address the limitations of the standard Merton Model und provide a comprehensive model for valuations of:
    - Single debt instruments with **different repayment agreements**
    - **Debt portfolios** irrespective of interest and repayment modalities
    - Risky debt with **continuous dividend payments**

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Different Repayment Agreements

Assumptions

- Firm has **two classes of claims**: debt and equity
- Firm has exactly **one debt instrument** in addition to its equity (with any kind of interest and principal repayment structure)
- Each payment to debtholders is **refinanced through new external capital** (either equity or debt)
- Changes in the value of the firm follow a **stationary random walk**
- Investors agree about the **volatility of assets** σ_V
- Firm pays **no dividends**

Extensions

- The **Merton (1974)** analysis can only be conducted if the firm holds only one zero-coupon debt
- **Geske (1977)** derives closed-form valuation for risky coupon bonds

- We generalize his coupon bond approach in order to be able to value risky debt for any kind of repayment form:
  - Lump sum repayment
  - Annuity repayment
  - Constant principal repayment

Merton formulas in their original form cannot be used to price the equity or the risky debt in this case!

Valuation Basics

At maturity T

$$E_T = \begin{cases} 0 & V_T \leq I_T + P_T \\ V_T - (I_T + P_T) & V_T > I_T + P_T \end{cases}$$

$$E_{(T-1)^+} = V_{T-1} \cdot N(h_1) - (I_T + P_T) \cdot e^{-r} \cdot N(h_2)$$

with $h_1 = \frac{ln\frac{V_{T-1}}{I_T+P_T}+r+\frac{\sigma_V^2}{2}}{\sigma_V}$ and $h_2 = h_1 - \sigma_V$.

Before maturity

$$E_{(T-1)^-} = \begin{cases} 0 & V_{T-1} \leq V_{T-1}^* \\ E_{(T-1)^+} - (I_{T-1} + P_{T-1}) & V_{T-1} > V_{T-1}^* \end{cases}$$

$$E_{T-1}(V_{T-1}^*) = I_{T-1} + P_{T-1}$$

V_{T-1}^* … killing price (i.e., bankruptcy trigger)

= critical value of the firm at $t = T - 1$ where the value of the equity at $(T-1)^-$ is just as large as the interest and principal payments that are due at $t = T - 1$.

Equity must now be interpreted as a compound option rather than a simple European call option

Valuation Basics

Equity holders have two options in each period t:

- Pay the interest and principal repayments
- Refuse to make the required payments and declare bankruptcy

$$E_{(T-s)^+} = V_{T-s} \cdot N_s(h_1^1, \ldots, h_1^s; \rho_s) - \sum_{t=0}^{s-1} (I_{T-t} + P_{T-t}) \cdot e^{-r\cdot(s-t)} \cdot N_{s-t}(h_2^1, \ldots, h_2^{s-t}; \rho_{s-t})$$

with $h_1^\tau = \frac{\ln\left(\frac{V_{T-s}}{V^*_{T-s+\tau}}\right) + \left(r + \frac{\sigma_V^2}{2}\right)\cdot\tau}{\sigma_V \cdot \sqrt{\tau}}$ and $h_2^\tau = h_1^\tau - \sigma_V \cdot \sqrt{\tau}$

$$D_0 = V_0 - E_0 = V_0[1 - N_T(d_1^1, \ldots, d_1^T; \rho_T)] + \sum_{t=1}^{T} (I_t + P_t) e^{-r\cdot t} \cdot N_t(d_2^1, \ldots, d_2^t; \rho_t)$$

Interpretation and Analysis

$$D_0 = \underbrace{\sum_{t=1}^{T}(I_t + P_t)\cdot e^{-r\cdot t}}_{\text{Value of risk-free bond}} - \underbrace{\sum_{t=1}^{T}\underbrace{\underbrace{\left[(I_t + P_t)\cdot e^{-r\cdot t} - V_0 \cdot \underbrace{\frac{N_{t-1}(d_1^1,\dots,d_1^{t-1};\rho_{t-1}) - N_t(d_1^1,\dots,d_1^t;\rho_t)}{N_{t-1}(d_2^1,\dots,d_2^{t-1};\rho_{t-1}) - N_t(d_2^1,\dots,d_2^t;\rho_t)}}_{\text{Risk-neutral expected discounted recovery rate given default at } t}\right]}_{\text{Discounted loss given default at } t} \cdot \underbrace{\left[N_{t-1}(d_2^1,\dots,d_2^{t-1};\rho_{t-1}) - N_t(d_2^1,\dots,d_2^t;\rho_t)\right]}_{\text{Risk-neutral total probability of default at } t}}_{\text{Expected discounted loss at } t}}_{\text{Present value}}$$

Risk Aversion

- Assumption of risk-neutrality rarely holds in practice as typical investors are risk-averse
  - Not willing to invest at **risk-free interest rate**
  - Require **compensation for bearing risk**

More interested in risk-adjusted yield than in the risk-neutral yield

- We therefor calculate:
  - **Risk-averse probabilities** (default probabilities) and **recovery rates**
  - **Risk-adjusted yields** using expected risk-averse cash flows

Multiple Debt Instruments

Valuation Setup

- In reality, a company often does not hold only a single debt instrument. Its debt rather consists of a portfolio of different instruments with varying repayment agreements.
- **Assumptions made:**
  - All debt securities will mature at T
  - All have the same rank
  - Nominal value of total debt Nom_t is determined by the nominal value of the specific debt instrument Nom_t^S and the miscellaneous debt instruments Nom_t^M

Claim of the creditor γ_t of the specific debt instrument depends on total debt

$$\gamma_t = \frac{Nom_{t-1}^S + I_t^S}{Nom_{t-1} + I_t}$$

Valuation Setup

Valuation of each specific debt instrument needs to be modified due to changing recovery rates in the event of bankruptcy

But Note: Valuation of equity and the determination of the killing prices, V_t^*, do not change

Default Clause Regulation:

- **International Default Clause:**
  Bondholders may call their bonds before maturity if the firm cannot serve payments on time on any of the other debt instruments issued
- **Cross Default Clause:**
  Creditor can demand the early termination of his own loan when the debtor defaults in another credit relationship

Valuation and Practical Implementation

$$D_T^S = \begin{Bmatrix} \gamma_T \cdot V_T & if\ V_T < Nom_{T-1}^S + I_T^S \\ Nom_{T-1}^S + I_T^S & if\ V_T \geq Nom_{T-1}^S + I_T^S \end{Bmatrix}$$

$$D_0^S = V_0 \left[\gamma_1 + \sum_{t=1}^{T-1} (\gamma_{t+1} - \gamma_t) \cdot N_t(d_1^1, \dots, d_1^t; \rho_t) - \gamma_T \cdot N_T(d_1^1, \dots, d_1^T; \rho_T) \right] + \sum_{t=1}^{T} (I_t^S + P_t^S) e^{-r \cdot t} \cdot N_t(d_2^1, \dots, d_2^t; \rho_t)$$

Data Sources	Implementation
• Plan balance sheet • Plan profit and loss statement • Interest and redemption schedule • Strategic investment plan • Financial plan • Forecasts of future interest rate levels	Straightforward if present value of total assets, V_0, and volatility of the assets, σ_V, are known
	If value of total assets, V_0, and volatility of the assets, σ_V, are unknown, calibration from market data using present value of the equity, E_0, and its corresponding volatility, σ_E

Numerical Example

Single Debt Instrument

Term T	5
Asset value V_0	100.00
Face value Nom_0	70.00
Risk-free rate r	2.00%
Asset volatility σ_V	15.00%
Asset beta β_V	1.00
Drift of the market of unlevered assets μ_M	4.00%

Form of Repayment	Lump Sum	Annuity	Constant Principal	Zero Coupon
Nominal interest rate p.a. i_{nom}	2.50%	2.50%	2.50%	-
Value of risk-free debt $D_0^{riskless}$	70.58	70.98	70.96	63.34
Value of risky debt D_0^{risky}	70.24	70.92	70.91	62.29
Expected risk-neutral continuous yield to maturity $E_0'(y_T)$	2.00%	2.00%	2.00%	2.00%
Expected risk-averse continuous yield to maturity $E_0(y_T)$	2.17%	2.01%	2.01%	2.17%

Multiple Debt Instruments

Form of Repayment Specific Debt	Lump Sum	Zero Coupon
Asset Value V_0	200.00	
Face value specific debt Nom_0^S	70.00	70.00
Nominal interest rate p.a. i_{nom}	2.50%	-
Share on total debt $\gamma_t = \gamma$	50.62%	49.38%
Value of riskless debt $D_0^{riskless}$	70.58	63.34
Value of risky debt D_0^{risky}	70.35	62.23
Expected risk-neutral continuous yield to maturity $E_0'(y_T)$	2.00%	2.00%
Expected risk-averse continuous yield to maturity $E_0(y_T)$	2.17%	2.16%

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Multiple Debt Instruments with Continuous Dividends

Valuation Setup

- Incorporation of **continuous dividends** with a constant return of q
- Dividends reduce the value of the firm (value of total assets)

$$\frac{dV_t}{V_t} = (\mu_V - q)dt + \sigma_V dz$$

- The equity holders receive continuous dividend payments of $q \cdot V_t$ from $t = 0$ until bankruptcy or $t = T$, whichever comes first.

The value of the firm ex dividend, V_0^{ex}, is the difference between the value of the firm without dividend payments, V_0, and the present value of the expected dividends.

Valuation

$$D_0^S = V_0^{ex}\left[\sum_{t=1}^{T-1}\gamma_t \cdot N_t(d_1^{1,ex},\dots,d_1^{t-1,ex},-d_1^{t,ex};\rho_t')\right] + \sum_{t=1}^{T}(I_t^S + P_t^S)e^{-r\cdot t}\cdot N_t\left(d_2^{1,ex},\dots,d_2^{t,ex};\rho_t\right)$$

$$= V_0\left[1-(1-e^{-q})\sum_{t=1}^{T}e^{-q(t-1)}\cdot N_{t-1}\left(d_1^{ex,1},\dots,d_1^{ex,t-1};\rho_{t-1}\right)\right]\cdot\left[\sum_{t=1}^{T-1}\gamma_t\cdot N_t(d_1^{1,ex},\dots,d_1^{t-1,ex},-d_1^{t,ex};\rho_t')\right] + \sum_{t=1}^{T}(I_t^S + P_t^S)e^{-r\cdot t}\cdot N_t\left(d_2^{1,ex},\dots,d_2^{t,ex};\rho_t\right)$$

Formula can be applied for any kind of repayment agreement, to debt portfolios and under consideration of continuous dividend payments

Continuous dividend payment rate q	0%	1%	2%	3%
Value of risk-free debt $D_0^{riskless}$	70.58	70.58	70.58	70.58
Value of risky debt D_0^{risky}	70.24	69.79	69.25	68.60

Conclusion

We provide three multivariate extensions of the Standard Merton model

1. Pricing risky debt irrespective of its interest and redemption payment structure
   - Provide repayment specific closed-form solutions as well as a generic formula
   - Provide formulas deriving risk-neutral and risk-averse probabilities of default as well as recovery rates
   - Provide insights for both risk-neutral and risk-averse investors
2. Valuing multiple debt instruments within the same company
   - Standard formulas cannot be used
   - Show the difference between single debt and multiple debt valuation
3. Pricing single or multiple debt instruments with continuous dividend payments
   - Provide formula for both single and multiple debt instruments within a company

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Appendix – Probabilities of Default

Cumulative Probabilities

Risk-neutral cumulative survival probability:

$$Prob'(No\ Default\ until\ t) = N_t(d_2^1, \ldots, d_2^t; \rho_t)$$

Risk-neutral cumulative default probability:

$$Prob'(Default\ until\ t) = 1 - N_t(d_2^1, \ldots, d_2^t; \rho_t)$$

Total Probabilities

Risk-neutral total default probability:

$$Prob'(No\ Default\ until\ t-1\ and\ Default\ at\ t) = N_{t-1}(d_2^1, \ldots, d_2^{t-1}, -d_2^t;\ \rho_t')$$

Conditional Probabilities

Risk-neutral conditional default probability:

$$Prob'(Default\ at\ t | No\ Default\ until\ t-1) = \frac{N_t(d_2^1, \ldots, d_2^t; \rho_t)}{N_{t-1}(d_2^1, \ldots, d_2^{t-1}; \rho_{t-1})}$$

Appendix – Multiple Debt Instruments

$$D_0^S = V_0 \left[\gamma_1 + \sum_{t=1}^{T-1} (\gamma_{t+1} - \gamma_t) \cdot N_t(d_1^1, \dots, d_1^t; \rho_t) - \gamma_T \cdot N_T(d_1^1, \dots, d_1^T; \rho_T) \right] + \sum_{t=1}^{T} (I_t^S + P_t^S) e^{-r \cdot t} \cdot N_t(d_2^1, \dots, d_2^t; \rho_t)$$

$$\rho_s = \langle \rho_{\tau_1,\tau_2} \rangle = \begin{cases} 1 & if\ \tau_1 = \tau_2, \tau_1 = 1, \dots, \tau, \tau_2 = 1, \dots, \tau \\ \sqrt{\tau_1 / \tau_2} & if\ \tau_1 < \tau_2, \tau_1 = 1, \dots, \tau, \tau_2 = 1, \dots, \tau \\ 0 & else \end{cases}$$

$$D_0^S = V_0 \sum_{t=1}^{T-1} \gamma_t \cdot N_t(d_1^1, \dots, d_1^{t-1}, -d_1^t;\ \rho_t') + \sum_{t=1}^{T} (I_t^S + P_t^S) e^{-r \cdot t} \cdot N_t(d_2^1, \dots, d_2^t; \rho_t)$$

$$\rho_t' = \langle \rho_{\tau_1,\tau_2}' \rangle = \begin{cases} 1 & if\ \tau_1 = \tau_2, \tau_1 = 1, \dots, \tau, \tau_2 = 1, \dots, \tau \\ \sqrt{\tau_1 / \tau_2} & if\ \tau_1 < \tau_2, \tau_1 = 1, \dots, \tau, \tau_2 = 1, \dots, \tau - 1 \\ -\sqrt{\tau_1 / \tau_2} & if\ \tau_2 = \tau, \tau_1 = 1, \dots, \tau \\ 0 & else \end{cases}$$